Cluster structures on Cox rings

(Arxiv... Luca Francone + Cox)

Intro: Z smooth irred variety $/\mathbb{C}$ $(k = \bar{k},\ \mathrm{char}\,k = 0)$
$\mathrm{Pic}(Z) \simeq \mathbb{Z}^n$

$$\mathrm{Cox}(Z) = \bigoplus_{[L] \in \mathrm{Pic}(Z)} \mathrm{Cox}(Z)_{[L]}$$

$$\mathrm{Cox}(Z)_{[L]} \simeq H^0(Z, L)$$

Ex: $Z = \mathbb{P}^n$ coordinates $[z_0 : \dots : z_n]$
$\mathrm{Cox}(Z) = \mathbb{C}[z_0, \dots, z_n]$
$\mathrm{Cox}(Z)_{[\mathcal{O}(d)]} = \mathbb{C}[z_0, \dots, z_n]_d$ $(d \geq 0)$

$\mathrm{Cox}(Z) \leftrightsquigarrow$ relevant info on bir geometry of Z.
(Hu - Keel: Z proj + Hp
all small bir models of Z
are VGIT of $T \curvearrowright \mathrm{Spec}(\mathrm{Cox}(Z))$.)

Pb: describe $\mathrm{Cox}(Z)$
- find generators, relations
- $\mathrm{Cox}(Z)$ is f. generated (challenging).

Today: $\mathrm{Cox}(Z) \overset{?}{=} \mathcal{U}$ (upper cluster algebra)

Ex:
- Construct bases
- Comb models for $\dim H^0(Z, L)$ (formulas)
- Categorification.

Ex.: $\mathrm{Cox}(Z) = \mathcal{U}$
1) $Gr(k,n) \longrightarrow$ F-Z (2002 For $k = 2$)
Scott (2006 general)
2) (Conjectured G-L-S) partial flag varieties (2008)
(open in most cases)
3) Work of G-H-K, M related to cluster varieties

Reminders on Upper cluster algebras Fix $\mathcal{F}/\mathbb{C}$

Def: A seed $t = (I, B, x)$ of $\mathcal{F}$ consists of

1) I vertices set (finite set)
$= I_{uf} \cup I_f$
(mutable) (frozen)

2) $B = \mathbb{Z}^{I \times I_{uf}}$ $\quad B = \begin{array}{c} I_{uf} \\ \left(\quad\right) \end{array} I$
exchange matrix

+ conditions

3) $x = (x_i)_{i \in I} \subseteq \mathcal{F}$
cluster
$\hookrightarrow$ alg indep over $\mathbb{C}$.

$$\mathcal{A}_t := \mathrm{Spec}\left(\mathbb{C}[x_i]_{i \in I_f}[x_k^{\pm 1}]_{I_{uf}}\right) \simeq \mathbb{A}^{I_f} \times (\mathbb{A} \setminus \{0\})^{I_{uf}}$$

Mutations: $k \in I_{uf}$ $\quad t \xrightarrow[\text{mutation at } k]{\mu_k} \mu_k(t)$

$\mu_k(t) = (\mu_k(I), \mu_k(B), \mu_k(x))$ def by

1) $\mu_k(I) = I$

2) $\mu_k(B) = \{$ rules

3)
$$\mu_k(x)_i = \begin{cases} x_i & i \neq k \\ \dfrac{\prod_{j \in I} x_j^{\max\{0, b_{jk}\}} + \prod_{j \in I} x_j^{\max\{0, -b_{jk}\}}}{x_k} & i = k \end{cases}$$

$$\mathcal{A}_t \overset{\mu_k}{\underset{\sim \text{bir}}{\dashrightarrow}} \mathcal{A}_{\mu_k(t)}$$

Iterate: s a seed of $\mathcal{F}$

$s \sim t$ if $s = \mu_{k_1} \circ \dots \circ \mu_{k_n}(t)$ $(k_i \in I_{uf})$

$$\mathcal{A}_t \overset{\mu_{st}}{\underset{\sim \text{bir}}{\dashrightarrow}} \mathcal{A}_s$$

Def:
$$\mathcal{O}\left(\bigcup_{s \sim t} \mathcal{A}_s \Big/ \sim_\mu\right) = \mathcal{U}(t)$$
upper cluster algebra
$\hookrightarrow$ cluster variety of t

Rks: 1) $s \sim t$ $\quad \mathcal{U}(s) = \mathcal{U}(t)$.

2) $\mathcal{U}(t) = \bigcap_{s \sim t} \mathcal{O}(\mathcal{A}_s) \subseteq \mathcal{F}$

3) (Laurent Phenomenon) is $s \sim t$ and z_i is a cluster variable of s, then $z_i \in \mathcal{U}(t)$.

4) The cluster variables satisfy relations given by the exchange matrices

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Back to Cox rings: Z $\quad \mathrm{Cox}(Z) = \mathcal{U}(\,?\,)$

Idea: induce from open subsets!

Setting: $Y \underset{\text{open}}{\subseteq} Z$ s.t.
1) Y affine
2) $\mathrm{Pic}(Y) = \{0\}$
3) $\mathcal{O}(Y)^\times = \mathbb{C}^\times$

$Z \setminus Y = \bigcup_{d \in D} E_d$ $\quad E_d$ irreducible components

Then
$$\bigoplus_{d \in D} \mathbb{Z} E_d \longrightarrow \mathrm{Pic}(Z)$$
$$E \longmapsto [\mathcal{O}(E)]$$
is an iso.

$$\bigoplus_{n \in \mathbb{Z}^D} H^0\left(Z, \mathcal{O}\left(\textstyle\sum n_d E_d\right)\right) = \mathrm{Cox}(Z)$$
ring structure on hom components: multiplication of rational fcts on Z.

We have a restriction map
$$\mathrm{Res}: \mathrm{Cox}(Z) \longrightarrow \mathcal{O}(Y)$$
homogenization
$$f \longmapsto f_{|Y}$$
$f \in H^0(Z, \mathcal{O}(\sum n_d E_d))$

Explained: $f \in \mathbb{C}(Z)$ s.t. $\mathrm{div}(f) + \sum n_d E_d \geq 0$
$\Rightarrow \mathrm{div}(f)_{|Y} \geq 0$
$\Rightarrow f_{|Y} \in \mathcal{O}(Y)$.

Homogenization Let $p \in \mathcal{O}(Y)$, define

$\uparrow p \in \mathrm{Cox}(Z)$

$$\uparrow p = p \in H^0\left(Z, \mathcal{O}\left(\sum_{d \in D} -\nu_{E_d}(p)\, E_d\right)\right)$$

$\uparrow p$ is homogeneous, $\mathrm{Res}(\uparrow p) = p$, and it is minimal with these properties

Ex: $Y = \{z_0 \neq 0\} \subseteq Z = \mathbb{P}^n$ coord $[z_0 : \dots : z_n]$
standard homogenization.

Thm: Assume $\mathcal{O}(Y) = \mathcal{U}(t)$ $\quad t = (I, B, x)$
(+ Mild Hp). There exists an explicit seed
$\uparrow t = (\uparrow I, \uparrow B, \uparrow x)$ s.t.

1) $\mathcal{U}(\uparrow t) \subseteq \mathrm{Cox}(Z)$
$\hookrightarrow$ graded subalgebra.

2) Any cluster variable of $\mathcal{U}(\uparrow t)$ is homogeneous.

3) $I_{uf} \subseteq (\uparrow I)_{uf}$ and $\forall\, i_1, \dots, i_n \in I_{uf}$
$I_f \subseteq (\uparrow I)_f$ $\quad k \in I$

$$\mathrm{Res}\left(\mu_{i_1} \circ \dots \circ \mu_{i_n}(\uparrow x)_k\right) = \mu_{i_1} \circ \dots \circ \mu_{i_n}(x)_k.$$

Moreover If $\tilde{t}$ is a seed s.t. $\mathcal{U}(\tilde{t}) = \mathrm{Cox}(Z)$ and $\mathcal{U}(\tilde{t})$ satisfies 2, and 3,
then $\tilde{t} = \uparrow t$.

Rks: 1) Explicit closed formulas for $\uparrow t$
2) In general $\mathcal{U}(\uparrow t) \subseteq \mathrm{Cox}(Z)$ may be strict.
3) $\mathcal{U}(\uparrow t)_M = \mathrm{Cox}(Z)_M$ for some (explicit $M \in \mathcal{U}(\uparrow t)$)
4) Criterion for studying if $=$

Examples: Below

Applications: 1) Geometric interpretation of a construction of G-L-S for partial flag varieties.

2) $\mathrm{Cox}(G/B)$ is an Upper cluster algebra
(G semisimple alg gp.)

3) Y cluster variety (finite type)
$\cap$
Z along partial compactification (Definition in the paper)

$\rightsquigarrow \mathrm{Cox}(Z)$ is an Upper cluster algebra

Ex : 1) $Y = \mathbb{A}^3 \hookrightarrow \mathbb{P}^3$ $\quad [Z_0 : Z_1 : Z_2 : Z_3]$ Homo coordinates

$(z_1, z_2, z_3) \longmapsto [1 : z_1 : z_1 : z_2]$

$\mathcal{O}(Y) = \mathcal{U}(t)$ where

$$t \qquad \begin{array}{ccccc} 1 & & 2 & & 3 \\ \circ & \longleftarrow & \circ & \longleftarrow & \blacksquare \end{array} \rightsquigarrow B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

cluster variables : $x_1 = z_1$, $\quad x_2 = z_2 z_1 - 1$, $\quad x_3 = z_1 z_2 z_3 - z_3 - z_1$

$$\mu_2(x)_2 \quad \frac{(z_1 z_2 z_3 - z_3 - z_1) + z_1}{z_2 z_1 - 1} = z_3$$

$$\uparrow t = \begin{array}{ccccc} 1 & & 2 & & 3 \\ \circ & \longleftarrow & \circ & \longleftarrow & \blacksquare \\ & \nwarrow\!\!\nwarrow & \Uparrow & & \\ & & \blacksquare\ 0 & & \end{array}$$

$\uparrow x_1 = Z_1 \qquad\qquad \uparrow x_0 = Z_0$

$\uparrow x_2 = Z_2 Z_1 - Z_0^2$

$\uparrow x_3 = Z_1 Z_2 Z_3 - Z_3 Z_0^2 - Z_1 Z_0^2$

2) $U = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{C} \right\} \qquad B^- = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \subset SL(3)$

$U \hookrightarrow B^- \backslash SL(3)$

$\mathcal{O}(U) = \mathcal{U}(t)$

$$t = \begin{array}{ccccc} & & \circ\, 1 & & \\ & \swarrow & & \nwarrow & \\ \blacksquare\ 3 & & & & \blacksquare\ 2 \end{array}$$

$x_1 = a$

$x_2 = ac - b$

$x_3 = b$

$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$

$$\uparrow t = \begin{array}{ccccc} \blacksquare\ \varpi_2 & & & & \blacksquare\ \varpi_1 \\ & \nwarrow & & \swarrow & \\ & & \circ\, 1 & & \\ & \swarrow & & \nwarrow & \\ \blacksquare\ 3 & & & & \blacksquare\ 2 \end{array}$$

$\uparrow x_1 = \Delta_{1,2}$, determinant row 1, col 2

$\uparrow x_2 = \Delta_{12,23}$

$\uparrow x_3 = \Delta_{1,3}$

$\uparrow x_{\omega_1} = \Delta_{1,1}$

$\uparrow x_{\omega_2} = \Delta_{12,12}$

3) Cluster structure on $\mathrm{Cox}(\overline{M}_{0,5})$ in the paper.